Math 1310
Section 5.4
Laws of Logarithms

The laws:
YOU MUST LEARN THESE!

If m, n and b are positive numbers, $b \neq 1$ then

1. $\log_b b = 1$ $\quad b^1 = b$

2. $\log_b 1 = 0$ $\quad b^0 = 1$

Product Rule

3. $\log_b (PQ) = \log_b P + \log_b Q$ $\quad \log_2 xy = \log_2 x + \log_2 y$

Quotient Rule

4. $\log_b \left(\frac{P}{Q}\right) = \log_b P - \log_b Q$ $\quad \log\left(\frac{3}{2}\right) = \log 3 - \log 2$

Note: $\log_b \frac{1}{Q} = -\log_b Q$ Why? $\quad \log_b 1 - \log_b Q$ $= 0 - \log_b Q$

Power Rule

5. $\log_b P^n = n \log_b P$ $\quad \log_2 3^5 = 5 \log_2 3$

Inverse Rule

6. $b^{\log_b P} = P$

7. $\log_b b^P = P$ $\quad \ln e^2 = 2$ $\quad e^{\ln 4} = 4$

8. $\log_a x = \frac{\log_b x}{\log_b a}$ (Change of base formula) $\quad \log_2 5$?

Note: this results in $\log_a x = \frac{\ln x}{\ln a}$.

Examples using these properties:

Errors to avoid:

$$\log_a(x+y) \neq \cancel{\log_a x + \log_a y}$$

$\log(xy) = \log x + \log y$

$$\frac{\log_a x}{\log_a y} \neq \cancel{\log_a x - \log_a y}$$

$$(\log_a x)^3 \neq 3\log_a x$$

$\log_a(x^3)$

$(\log_a x)^3$

Using the properties to simplify logarithms

Example 1: Simplify each logarithm

$\log_2(32) = \log_2(2^5) = 5$ Rule #7

$\log_6 9 + \log_6 4 = \log_6(9\cdot 4) = \log_6(36) = \log_6(6^2) = 2$

product rule Rule 7

$\log_4(96) - \log_4(6) = \log_4\left(\frac{96}{6}\right) = \log_4(16) = \log_4(4^2) = 2$

$\log_3\left(\frac{1}{81}\right) = -\log_3(81) = -\log_3(3^4)$

$= -4$

$\log_2(4^6) = 6\log_2 4 = 6\log_2(2^2)$

power

$= 6\cdot 2 = 12$

$\log_b(\sqrt{b}) = \log_b(b^{1/2}) = 1/2$

$\log_a\left(\frac{1}{a^3}\right) - \log_a(a^2) = \log_a 1 - \log_a a^3 - \log_a a^2$

$0 - 3 - 2 = -5$

Separating One Complicated Logarithmic Expression:

Example 2: Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$\log_3(x(x+4))$ $= \log_3(x) + \log_3(x+4)$

Example 3: Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$\log\frac{3}{x}$ $\qquad\qquad$ $\log(3x) = \log 3 + \log x$

$= \log 3 - \log x$

Example 4: Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$$\ln\left(\frac{ab^3}{c^2d}\right) = \ln(ab^3) - \ln(c^2d)$$

$$= [\ln(a) + \ln(b^3)] - [\ln(c^2) + \ln d]$$

$$= [\ln(a) + 3\ln(b)] - [2\ln c + \ln d]$$

$$= \ln a + 3\ln b - 2\ln c - \ln d$$

Example 5: Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$$\log_3\left(\frac{x+5}{x^2-4}\right) = \log_3(x+5) - \log_3(x^2-4)$$

$$(x^2-4) = (x+2)(x-2)$$

$$= \log_3(x+5) - \log_3[(x+2)(x-2)]$$

$$= \log_3(x+5) - \log_3(x+2) - \log_3(x-2)$$

Example 6: Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$$\log_{10}\left[\frac{x^2(x+1)}{(x-3)(x+7)^4}\right] = \log[x^2(x+1)] - \log[(x-3)(x+7)^4]$$

$$= [\log(x^2) + \log(x+1)] - [\log(x-3) + \log((x+7)^4)]$$

$$= [2\log x + \log(x+1)] - [\log(x-3) + 4\log(x+7)]$$

$$= 2\log x + \log(x+1) - \log(x-3) - 4\log(x+7)$$

<u>Example 7:</u> Rewrite the expression in a form with no logarithm of a product, power, or quotient.

$\log_7\left(\frac{\sqrt{x+1}}{x^3}\right)$ $= \log_7(\sqrt{x+1}) - \log_7(x^3)$

$= \log_7((x+1)^{1/2}) - \log(x^3)$

$= \frac{1}{2}\log_7(x+1) - 3\log_7 x$

<u>Example 8</u>: Rewrite the following so that each logarithm contains a prime number.

$\log_2 35$ $= \log_2(5\cdot 7) = \log_2 5 + \log_2 7$

$\log_3 100$ $= \log_3(2^2 \times 5^2) = \log_3(2^2) + \log_3(5^2)$

$= 2\log_3 2 + 2\log_3 5$

$100 = 10 \times 10$

$\Rightarrow 2\times 5 \times 2 \times 5 = 2^2 \times 5^2$

Combining a sum of logarithmic expressions:

<u>Example 9:</u> Rewrite as a single logarithm.

$\log_3 x + \log_3 2$ $= \log_3(x\cdot 2) = \log_3(2x)$

Example 10: Rewrite as a single logarithm.

$$\log\left(x^2-16\right)-\log(x+4) = \log\left[\frac{(x^2-16)}{(x+4)}\right]$$

$$= \log\left[\frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}}\right]$$

$$= \log(x-4)$$

Example 11: Rewrite as a single logarithm.

$\ln x^3 = 3\ln x$

$$2\ln x - 5\ln(x+1) + \tfrac{1}{2}\ln(x-3)$$

$$= \ln(x^2) - \ln\left[(x+1)^5\right] + \ln\left[(x-3)^{1/2}\right]$$

$$= \ln(x^2) - \ln\left[(x+1)^5\right] + \ln\left(\sqrt{x-3}\right)$$

$$= \ln\left[\frac{x^2}{(x+1)^5}\right] + \ln\left(\sqrt{x-3}\right)$$

$$\ln\left[\frac{x^2}{(x+1)^5}\cdot\sqrt{x-3}\right]$$

$$\ln\left[\frac{x^2\sqrt{x-3}}{(x+1)^5}\right]$$

Example 12: Rewrite as a single logarithm.

$$3\log_5(x+2) - 2\log_5(x-1) - 2\log_5(x-7)$$

$$= \log_5\left[(x+2)^3\right] - \log_5\left[(x-1)^2\right] - \log_5(x-7)^2$$

$$= \log_5\left[(x+2)^3\right] - \left[\log_5\left[(x-1)^2\right] + \log_5\left((x-7)^2\right)\right]$$

$$\log_5\left[(x+2)^3\right] - \left[\log_5\left[(x-1)^2(x-7)^2\right]\right]$$

$$\log_5\left[\frac{(x+2)^3}{(x-1)^2(x-7)^2}\right]$$

Example 13: Rewrite as a single logarithm.

$$\underbrace{\ln\left(A^5\right) + \ln\left(A^3\right)} - \ln\left(A^6\right)$$

$$= \ln\left[A^5 \cdot A^3\right] - \ln\left(A^6\right)$$

$$= \ln\left(A^8\right) - \ln\left(A^6\right)$$

$$= \ln\left[\frac{A^8}{A^6}\right] = \ln\left(A^2\right)$$

$$= 2\ln A$$

The change of base formula:

Why? The change of base formula is typically used in three situations

- When an expression or equation involves logs in two or more different bases
- When we want to evaluate a logarithm on a calculator and need to convert a base to base 10 or base "e" to use our calculator.
- Many more advanced formulas in math are given in terms of the natural logarithm, so if our equation is in a different base, we need to change it to the natural logarithm to use the formula.

Example 14: Use the change of base formula to change $\log_2 17$ to natural log.

$$\log_2 17 = \boxed{\frac{\ln 17}{\ln 2}}$$

Example 15: Use the change of base formula to change $\log_7 12$ to base 10.

$$\log_7 12 = \boxed{\frac{\log 12}{\log 7}}$$